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ON THE COMPUTATION OF THE CONSTANT C_2 OF PLANCK'S EQUATION BY AN EXTENSION OF PASCHEN'S METHOD OF EQUAL ORDINATES

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1. INTRODUCTION

Planck's equation for the intensity of radiation J, of wave length λ, from a black body at the absolute temperature θ, namely

$$J = \frac{c_1}{\lambda^5\left(e^{\frac{c_2}{\lambda\theta}} - 1\right)} \tag{1}$$

or

$$J = c_1\lambda^{-5}\left(1 + e^{-\frac{c_2}{\lambda\theta}} + e^{-2\frac{c_2}{\lambda\theta}} + \ldots\ldots \text{etc.}\right)e^{-\frac{c_2}{\lambda\theta}} \tag{2}$$

appears to represent the results of all known observations, nearly or quite within the limits of experimental error. As $\lambda\theta$ decreases, the factor in the parenthesis in equation (2) approaches unity, and the equation tends toward the simpler equation of Wien, or

$$J = c_1\lambda^{-5}e^{-\frac{c_2}{\lambda\theta}} \tag{3}$$

If λ is expressed in microns and θ in centigrade degrees of the standard gas scale, the value of the constant c_2 is in the vicinity of 14400, being apparently between 14600 and 14200. If $\lambda = 0.75\mu$, which is about the limit of the visible red, the value of the factor in the parenthesis in equation (2) is about 1.00007 at the melting point of platinum; about 1.006 at the temperature of the arc; and about 1.04 at the apparent temperature of the sun. For decreasing wave lengths, these values approach unity rapidly. For any value of $\lambda\theta$ less than 3000, the value of the factor in the

parenthesis is less than 1.01. For visual observations on bodies at temperatures up to that of the arc, the use of Wien's equation as a sufficient approximation to Planck's equation is therefore entirely permissible, and our optical pyrometers are in fact designed on the assumption of the validity of equation (3).

If J_1 and J_2 are the intensities observed, either visually or otherwise, at wave lengths λ_1 and λ_2, when sighting on black bodies at the absolute temperatures θ_1 and θ_2, equation (3) gives us

$$\log \frac{J_1}{J_2} + 5 \log \frac{\lambda_1}{\lambda_2} = c_2 \left[\frac{1}{\lambda_2 \theta_2} - \frac{1}{\lambda_1 \theta_1} \right] \tag{4}$$

If the observations are made at a fixed wave length λ, as through monochromatic absorption glasses or with a spectrophotometer, equation (4) may be reduced to the form

$$\frac{1}{\theta_2} = \frac{1}{\theta_1} + \frac{\lambda}{c_2} \log \frac{J_1}{J_2} \tag{5}$$

by which an unknown temperature θ_2 may be determined from a known temperature θ_1 by observations of J_1 and J_2, if λ and c_2 are known. An instrument for making these observations conveniently at a known wave length in the visible spectrum constitutes an optical pyrometer.

If Wien's equation, together with a certain value of c_2, be assumed, an optical scale of temperature is thereby defined, when a numerical value has been assigned to some fixed reference temperature θ_1. Naturally, however, we desire to adopt such a value of c_2 as shall make this scale agree with the standard gas scale within their common range of about 650° C to 1650° C. The use of the optical scale above the limits of accurate work with the gas thermometer is frequently spoken of as an extrapolation of the gas scale, but it conduces to clearness to regard the optical scale as an independent one which can be and, for convenience, is adjusted to agreement with the gas scale where the two overlap. The accuracy of this adjustment depends on the value assigned to the constant c_2, which is therefore of importance for optical pyrometry in addition to its purely scientific interest.

From equation (4) it is evident that any two observations of the intensity J, at known temperatures and wave lengths, determine the value which c_2 must have in order that equation (3) shall be satisfied. If the observations are made at a fixed wave length and if $\lambda\theta$ is small enough that equations (3) and (2) are sensibly identical, equation (5) or any equivalent equation may be used to find the value of c_2, and the most exact determinations of c_2 have been made in this way, by observations of the so-called isochromatic curves of intensity as a function of temperature with λ constant.

The other most obvious method of determining c_2 is to make the observations at various wave lengths on a body kept at a fixed temperature, i. e., to observe an isothermal or "energy curve" as it is usually called. Virtually all our methods of analyzing the numerical results of observations on black-body radiation are due to Paschen. One valuable method of determining the value of c_2 from an observed energy curve, devised by Paschen[1] before the publication of either Wien's or Planck's equations, was in substance based on the supposition that the equation

$$J = c_1\lambda^{-a} e^{-\frac{c_2}{\lambda\theta}} \tag{6}$$

of which Wien's equation is a special case, was generally valid. Since the method involves the use of wave lengths so large that Wien's equation is sensibly in error, it does not give constant values of c_2 when used on an accurately observed energy curve, as has been noted by Coblentz;[2] but the method may be so modified as to be applicable to a curve represented by Planck's equation, as will be shown in this paper.

2. THE RELATION BETWEEN THE VALUES OF C_2 AND $\lambda_m\theta$

From equation (3) we get

$$\log J = \log c_1 - 5 \log \lambda - \frac{c_2}{\lambda\theta} \tag{7}$$

and

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[1] Ann. d. Phys. **58**, 455; 1896. Ann. d. Phys. **60**, 662; 1897.

[2] Phys. Rev. **31**, p. 317, Sept., 1910.

$$\frac{d \log J}{d\lambda} = -\frac{5}{\lambda} + \frac{c_2}{\lambda^2\theta}$$

If λ_m is the wave length at which J is a maximum, we therefore have

$$c_2 = 5\lambda_m\theta \tag{8}$$

If Wien's equation were exact and if the value of λ_m could be precisely determined, for the given temperature, equation (8) would determine the value of c_2. This method, however, can not give accurate results if λ_m has to be determined merely by inspection of the energy curve, because the abscissa of the highest point of the rounded top of the curve can not be determined with any great precision.

Now let a horizontal secant be drawn across the energy curve somewhere below the maximum, and let λ_1 and λ_2 be the wave lengths of the two points of intersection. These values may be read off quite exactly if the secant is drawn at an appropriate height. Equation (7) applied to two *equal* values of J then gives us

$$5 \log \lambda_1 + \frac{c_2}{\lambda_1\theta} = 5 \log \lambda_2 + \frac{c_2}{\lambda_2\theta}$$

whence

$$c_2 = 5\ \theta \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1} (\log \lambda_2 - \log \lambda_1). \tag{9}$$

The application of equation (9) to an observed energy curve constitutes Paschen's method referred to above. If the observed curve were really represented by Wien's equation, equation (9) would give the value of c_2 and, by equation (8), of λ_m, much more accurately than mere inspection of the curve. We have to find how the process must be modified to allow for the fact, shown by equations (2) and (3), that for given values of c_1 and c_2 Planck's curve is everywhere higher than Wien's curve, and that this difference of ordinates increases with the wave length at which the curves are compared.

Let us first find the relation which is satisfied at the maximum ordinate of Planck's curve. By differentiating equation (1) we get

$$\frac{dJ}{d\lambda} = \frac{c_1\lambda^{-5}}{\left(e^{\frac{c_2}{\lambda\theta}} - 1\right)^2} \frac{c_2}{\lambda^2\theta} e^{\frac{c_2}{\lambda\theta}} - \frac{5c_1\lambda^{-6}}{e^{\frac{c_2}{\lambda\theta}} - 1}$$

and the condition that J shall be a maximum enables us to reduce this to the form

$$c_2 = 5\ \lambda_m \theta \left[1 - e^{-\frac{c_2}{\lambda_m \theta}}\right] \qquad (10)$$

The exponential term is small and we may proceed by successive approximations. Disregarding the correction and setting $\frac{c_2}{\lambda_m \theta} = 5$ we have $e^{-5} = 0.00674$, whence as a second approximation, $\frac{c_2}{\lambda_m \theta} = 4.9663$. A third approximation gives 4.9651, and a fourth does not change these figures. We therefore have for a curve which can be represented by Planck's equation

$$c_2 = 4.965\ \lambda_m \theta \qquad (11)$$

If Planck's equation is a correct representation of the observed energy curve, we can therefore find c_2 from equation (11) by inspection of the curve; but this method is open to the same objection as in the case of Wien's equation, and it is not possible to derive from Planck's equation any exact relation as simple as Paschen's equation (9), by which we may make use of the wave lengths for any pair of equal ordinates. Nevertheless, the importance of the value of the constant c_2 makes it desirable to utilize energy curves as well as isochromatics in determining this value, and to use a more accurate method than that of determining λ_m by inspection.

In the following sections two methods are given for finding the value of c_2 by the method of equal ordinates from an observed energy curve, when the portions of the curve used are representable by Planck's equation. The first of these methods, which is only approximate, is indirect and consists in the substitution in equation (9) of corrected values of the wave lengths read from the energy curve, the point of the method being in the determination of these corrections. It occurred to one of the writers in consequence of a conversation with Dr. W. W. Coblentz, who has already alluded to it.[3]

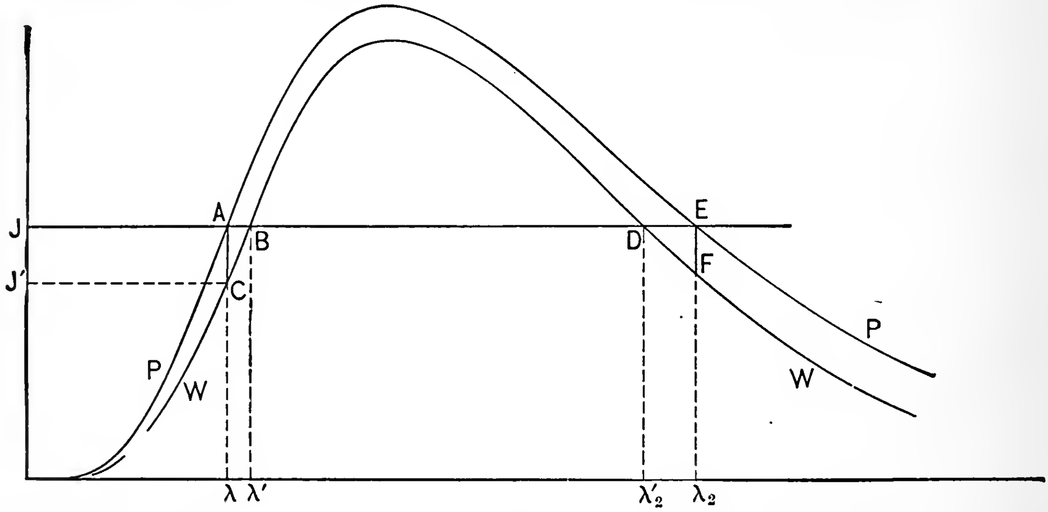
The second method is based directly upon Planck's equation, and gives a relation similar to equation (9) but with certain correction terms added. These may be computed exactly, but the

[3] Coblentz, loc. cit. See also Jahrb. der Radioaktivität und Elektronik 8, p. 1; 1911.

work may be shortened by an approximation which will be shown to be permissible. This method, which is due to the second of the writers, is simpler in application; at least as exact, even in its approximate form; and altogether preferable to the first method, which retains a value only for checking purposes. Each of the methods presupposes a knowledge of the approximate value of c_2, so that in some cases successive approximations might be necessary.

3. COMPUTATION OF WIEN'S CURVE FROM AN OBSERVED CURVE

We now assume, as is justified by the experimental confirmations of the validity of Planck's equation, that if the results of accurate observations on $J=f(\lambda)$ at θ constant are plotted, the



resulting curve may be represented by Planck's equation with appropriate values of c_1 and c_2. The value of c_1 influences the vertical scale but not the form of the curve, and need not occupy our attention.

Let *PP* in the figure be the observed curve, and let *WW* be a curve plotted from Wien's equation with the same values of c_1 and c_2 that must be used to make Planck's equation represent the observed curve. Let the horizontal secant *ABDE* cut the curves at the wave lengths λ λ' and λ'_2 λ_2. Let J be the ordinate of A and J' the ordinate of a point C on the Wien curve directly below A.

The curves are not far apart and if the secant is not too close to the maximum of the inner curve W, the lines BC and DF

may be treated as straight. Considering the triangle ABC, we now have

$$\frac{CA}{AB} = \frac{dJ'}{d\lambda} = \frac{J - J'}{\lambda' - \lambda}$$

whence

$$\lambda' - \lambda = \frac{J - J'}{\frac{dJ'}{d\lambda}} \tag{12}$$

From equations (1) and (3) we have

$$J - J' = c_1 \lambda^{-5} e^{-\frac{c_2}{\lambda\theta}} \left(e^{\frac{c_2}{\lambda\theta}} - 1 \right)^{-1}$$

From equation (3) we get

$$\frac{dJ'}{d\lambda} = c_1 \lambda^{-6} e^{-\frac{c_2}{\lambda\theta}} \left[\frac{c_2}{\lambda\theta} - 5 \right]$$

Substituting these values in equation (12), we have

$$\lambda' - \lambda = \frac{\lambda^2 \theta}{\left(e^{\frac{c_2}{\lambda\theta}} - 1 \right) (c_2 - 5\lambda\theta)}$$

whence

$$\lambda' = \lambda \left[1 + \frac{\lambda\theta}{\left(e^{\frac{c_2}{\lambda\theta}} - 1 \right) (c_2 - 5\lambda\theta)} \right] \tag{13}$$

If c_2 and θ are given, and λ is the wave length of an observed point A, equation (13) permits us to find the wave length λ' of the adjacent point B at the same height on Wien's curve. The same formula holds for the points D and E on the long wave length side of the maximum, the sign of the correction term changing when $5\lambda\theta$ passes through the value c_2. This correction term is small and an approximate value of c_2 is sufficient for computing its value unless $c_2 - 5\lambda\theta$ is small, or λ very large, and in either case the whole reasoning is invalidated because BC and DF are then no longer sensibly straight.

We have now, in equation (13), an approximate rule by which, from the wave lengths λ_1 and λ_2 at which equal intensities J were observed, the values λ'_1 and λ'_2 may be found for two points at

the same height on a curve drawn from Wien's equation with the values of c_1 and c_2, which would make Planck's equation fit the observed curve.

For example: Let $c_2 = 14600$; $\theta = 1460°$abs. $= 1187°$C; $\frac{c_2}{\theta} = 10$.

The maximum of the Wien curve lies at $\lambda_m = c_2/5\theta = 2$, so that values of λ in the immediate vicinity of 2 must not be used. By equation (13) we get

For $\lambda =$	1	3	5	8
$\frac{\lambda'}{\lambda} =$	1.0000	0.9773	0.9478	0.8921

In general, the value of λ'/λ on the short wave length side of the maximum will be so close to unity that no correction is needed and only the other point need be considered. For a given temperature, the value of λ'/λ may be computed once for all, as is done above for 1187° C, and plotted against λ, so that the correction factor may be read off from this correction curve without a special computation for each point. A series of such curves constructed for different temperatures could be used by interpolation for intermediate temperatures.

Accordingly, if we read from an observed energy curve the wave lengths λ_1 and λ_2 at which the curve is intersected by a horizontal secant, equation (13) permits us to find the wave lengths λ_1' and λ'_2 which would satisfy Wien's equation with the same values of c_1 and c_2 as make Planck's equation fit the observed curve. Substitution of these values in Paschen's equation (9) will then give us the value of c_2, though λ_m for the observed curve is larger than λ'_m for the Wien curve, in the ratio 5:4.965.

4. METHOD OF COMPUTATION INDEPENDENT OF WIEN'S CURVE

If Planck's equation be written in the form

$$J = \frac{c_1}{\lambda^5 e^{\frac{c_2}{\lambda\theta}}\left(1 - e^{-\frac{c_2}{\lambda\theta}}\right)} \tag{14}$$

we have

$$\log J = \log c_1 - 5 \log \lambda - \frac{c_2}{\lambda\theta} - \log\left(1 - e^{-\frac{c_2}{\lambda\theta}}\right) \tag{15}$$

and if, as before, a horizontal secant at the height J intersects the observed energy curve at the wave lengths λ_1 and λ_2, equation (15) gives us

$$c_2 = 5\theta\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\Big[(\log\lambda_2 - \log\lambda_1) + \frac{1}{5}\log\left(1-e^{-\frac{c_2}{\lambda_2\theta}}\right) - \frac{1}{5}\log\left(1-e^{-\frac{c_2}{\lambda_1\theta}}\right)\Big] \tag{16}$$

[4]

Upon comparison with equation (9) it will be seen that the present equation differs from the former by two correction terms of opposite sign. By means of the formula [5]

$$\log\left(1-e^{-\frac{c_2}{\lambda\theta}}\right) = -e^{-\frac{c_2}{\lambda\theta}} - \frac{1}{2}e^{-\frac{2c_2}{\lambda\theta}} - \frac{1}{3}e^{-\frac{3c_2}{\lambda\theta}} - \dots \text{ etc.} \tag{17}$$

equation (16) may also be put into the form

$$c_2 = 5\theta\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\Big[(\log\lambda_2 - \log\lambda_1) - \frac{1}{5}\left(e^{-\frac{c_2}{\lambda_2\theta}} - e^{-\frac{c_2}{\lambda_1\theta}}\right) - \frac{1}{10}\left(e^{-\frac{2c_2}{\lambda_2\theta}} - e^{-\frac{2c_2}{\lambda_1\theta}}\right) + \dots \text{ etc.}\Big] \tag{18}$$

If in (18) all the correction terms after the first are neglected, as will be shown to be permissible, equation (18) enables us to compute c_2 from a preliminary approximate value much more conveniently than was possible by the method described in section 3.

We have now to investigate the error caused by neglecting all the correction terms after the first in equation (18). The error is evidently of the order of the first correction term neglected, inasmuch as the series in equation (17) is convergent for all possible values of λ and θ; the fractional error is therefore of the order of $\dfrac{\frac{1}{10}\left(e^{-\frac{2c_2}{\lambda_2\theta}} - e^{-\frac{2c_2}{\lambda_1\theta}}\right)}{(\log\lambda_2 - \log\lambda_1)}$. This expression may be simplified,

[4] An equivalent of this form of the complete solution for c_2 has been worked out and used by W. W. Coblentz. In rather complicated form, the solution was given in the Physical Review **29**, 553; 1909. Slight errors which appear in the formula as there printed are corrected in a paper by Dr. Coblentz, to appear in the Physical Review **32**, June–July, 1911.

[5] Peirce's "Short Table of Integrals," formula 768.

since λ_1 is less than λ_2, and in practical cases the second term of the numerator is accordingly small relative to the first, and since, furthermore, it is sufficiently accurate *for this purpose* to set $\frac{c_2}{\theta}=5\lambda_m$. Hence we may state the fractional error to be of the order of

$$\Delta=\frac{e^{-10\frac{\lambda_m}{\lambda_2}}}{10\log\frac{\lambda_2}{\lambda_1}} \tag{19}$$

The complete investigation of this expression for the error would require differentiation with regard to λ_1 or λ_2, and this leads to complicated expressions, but it will be sufficient to compute the magnitude of Δ for a few values of $\frac{\lambda_2}{\lambda_m}$. The values computed are as follows:

For $\frac{\lambda_2}{\lambda_m}=$	1.25	1.5	2.	3.	4.
$\Delta=$	0.00008	0.00017	0.00055	0.0019	0.0036

The values of λ_1 used in the computations were read from a curve representing Planck's equation. It should be noted that the expression for the error, in (19), does not contain the temperature θ, but is determined entirely by $\frac{\lambda_2}{\lambda_m}$ and $\frac{\lambda_2}{\lambda_1}$. Accordingly the error is not different for the curves of different temperatures if we take λ_2 at "corresponding points"[6] of the curves in the sense of Wien's displacement law, i. e., keep the ratio $\frac{\lambda_2}{\lambda_m}$ constant. This does not require that the same λ_2 be taken in computing c_2 from observed energy curves of different temperatures, which would be impracticable on account of the absorption bands not falling at "corresponding points." This discussion relates only to the equality of the errors at "corresponding points."

From a consideration of the errors involved we find that for most practical cases equation (18) may be simplified to the form:

$$c_2=\frac{\theta\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[5\ (\log\lambda_2-\log\lambda_1)-e^{-\frac{c_2}{\lambda_2\theta}}\right] \tag{20}$$

[6] See the note at the end of this paper.

which is only slightly more complicated than Paschen's equation itself. The error in this equation is of the same order as that given in (19), viz, less than 3 parts in 1000 for values of λ_2 up to about 3.5 times λ_m. For $\lambda_2 > 3.5\ \lambda_m$ the curve becomes so flat that no accurate readings could be made and the increase of the error beyond this point is of no importance. The best part of the curve to use is that for which λ_2 lies between $1.2\ \lambda_m$ and $2.5\ \lambda_m$, if the temperature is such that no absorption bands affect the experimental determination of this part of the curve.

If the experimental accuracy makes it desirable, the second correction term in equation (18) may be used or the exact equation (16) may be used, but it is doubtful if this will ever be necessary with the present experimental accuracy. In the same way as mentioned above in the discussion of equation (13) the correction to Paschen's equation can be calculated once for all from equation (16) and plotted for future reference. This is, however, hardly necessary, for it would be scarcely any less labor to read the correction from the correction curve than to calculate the exponential term in equation (20) by a slide rule and a short table of exponentials.

5. APPLICATION OF THE FOREGOING RESULTS

The application of the methods developed in this paper may be illustrated by a table of results obtained by readings from an energy curve drawn to represent Planck's equation with the values $\theta = 1442°$ abs. and $c_2 = 14500$, and having its maximum ordinate at $\lambda_m = 2\ \mu$. This curve, which was drawn several years ago, is on a rather small scale, viz, $1\ \mu = 20$ mm and $J_m = 243$ mm, so that the errors of *reading* from the curve are larger than they would usually be in practice. The readings were made in the vicinity of some of the 14 computed points ($\lambda = 0.5\ \mu$ to $\lambda = 7.8\ \mu$) from which the curve was constructed. They should give the exact values of λ_1 and λ_2 corresponding to the assumed data, $c_2 = 14500$ and $\theta = 1442°$ abs., within about 0.02 μ, except for the last value of λ_2 which may be more in error. On the whole, the accidental errors in the values of λ_1 and λ_2 as read are not unlike what might be expected in the case of an observed energy curve. In order not to give the computations an unfair advantage, the

value $c_2 = 14600$ was used as an approximation in equations (13) and (20).

The first and second columns of the table give the pairs of values of λ_1 and λ_2. The third column gives the values of c_2 computed by the Paschen equation (9); the fourth gives the values computed by the method of section 3, using equations (13) and (9); and the fifth those computed by equation (20). The last column gives the values of $(\lambda_2 - \lambda'_2)$ as found by equation (13) and shows the increasing horizontal separation of the Wien and Planck curves with increasing wave length.

1	2	3	4	5	6
λ_1	λ_2	c_2 equ. (9)	c_2 equs. (13) & (9)	c_2 equ. (20)	$\lambda_2 - \lambda'_2$
1.54	2.74	14610	14510	14480	0.053
1.30	3.45	14650	14500	14520	0.102
1.17	4.03	14700	14490	14500	0.143
1.03	4.95	14720	14440	14480	0.249
0.85	7.22	14860	14390	14520	0.655

The progressive increase of the values in column 3 shows clearly that Paschen's equation is not applicable to the present curve, drawn from Planck's equation. The values in column 4 are constant within the errors of λ_1 and λ_2 up to about 4.5 μ, or, to $\frac{\lambda_2}{\lambda_m} = 2.25$. Beyond this point they decrease, showing that the assumptions on which the method of section 3 is based are no longer sufficiently exact. The values in column 5 are constant within the errors of λ_1 and λ_2, and give the mean value 14500, showing incidentally that $c_2 = 14600$ was a sufficiently accurate value for use in the computations and that no further approximation is needed unless the values of λ_1 and λ_2 are much more accurate.

6. NOTE ON "CORRESPONDING POINTS"

Wien's displacement law states that if the curve of energy distribution in the spectrum of the radiation from a black body at the

absolute temperature θ be plotted with the intensity J as ordinate against the wave length λ as abscissa, the curve for any other temperature θ' may be obtained from the first by: (*a*) multiplying the wave length of each point by $\frac{\theta}{\theta'}$, and (*b*) changing the ordinate of each point in a ratio which is the same for all points but is not specified by this law.

Each point on the first curve is therefore represented by a "corresponding" point on the second. Since $\lambda' = \lambda\frac{\theta}{\theta'}$ or $\lambda'\theta' = \lambda\theta$, we may also define corresponding points on the energy curves for different temperatures, as points at which the value of $\lambda\theta$ is the same. If λ_1, λ_2, and λ_m are the abscissas of any three points on the curve for the temperature θ, and λ'_1, λ'_2, and λ'_m are the abscissas of the corresponding points on that for θ', we evidently have $\frac{\lambda'_m}{\lambda'_1} = \frac{\lambda_m}{\lambda_1}$; $\frac{\lambda'_m}{\lambda'_2} = \frac{\lambda_m}{\lambda_2}$; and $\frac{\lambda'_2}{\lambda'_1} = \frac{\lambda_2}{\lambda_1}$ [see equation (19)]. If λ_m is the abscissa of some point, such as a maximum of J, defined by some analytical condition, corresponding points may also be defined as points of which the wave lengths bear the same ratio to the wave length λ_m on the two curves.

Wien's displacement law makes no reference to the *form* of the function $J = f(\lambda)$; it merely says that any two energy curves for black body radiation may be made identical by plotting wave lengths in units which are proportional to the absolute temperature of the black body emitting the radiation, and then changing the scale of the ordinate J. The change of vertical scale is given by the Stefan-Boltzmann law which states that the area, $J_0 = \int_0^\infty J d\lambda$, of the curve is proportional to θ^4. This requires that the ordinates of corresponding points on two energy curves shall be in the ratio

$$\frac{J'}{J} = \frac{\theta'^5}{\theta^5}$$

Perhaps the clearest representation of the substance of these two laws is by a surface with λ, $J^{\frac{1}{5}}$, and θ as its x, y, and z coordinates. As this surface is swept out by the energy curve, in the

form $J^{\frac{1}{5}} = \phi(\lambda)$, with changing θ, any point on the energy curve describes a curve which lies in a plane containing the λ axis. All the points on this curve "correspond" to one another. The projection of this curve on the λ, θ plane is an equilateral hyperbola since $\lambda\theta$ remains constant.

WASHINGTON, April 26, 1911.

Fig. 1.—*The Voltmeters*